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THE MIXED SYMMETRICAL TEMPERATURE PROBLEM FOR A TRANSVERSELY ISOTROPIC ELASTIC LAYER†

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The problem of the uniform heating of a symmetrical three-layer plate with absolutely rigid outer layers, deformed solely due to thermal expansion, is solved. The generalized plane temperature problem is reduced to determining the stress-strain state, which is symmetrical with respect to two coordinates, of the inner layer (a soft filler) of transversely isotropic material using the equations of the mixed problem of elasticity theory. The layers are in ideal mutual contact. The conditions at the ends of the filler boundary layer (a thin contact layer) are not specified. On the remaining part of the ends of the filler the boundary conditions correspond to a free boundary. The problem has a finite smooth solution. The construct the exact solution a modification of Mathieu's method [1] is proposed, which consists of the fact that, in addition to ordinary Fourier series, solutions in polynomials are used. It is shown that the presence of these solutions in polynomials enables the convergence of the Fourier series to be accelerated considerably. © 2000 Elsevier Science Ltd. All rights reserved.

The problem of elasticity theory for a rectangular layer was considered previously in [2], where the case of the first fundamental symmetrical problem for a layer is investigated in detail.

1. METHOD OF SOLUTION

The problem of the uniform heating of a symmetrical three-layer plate with absolutely rigid outer layers, undergoing thermal expansion, is solved. It is assumed that the inner layer (the filler) of transversely isotropic material has practically no effect on the outer layers, in view of its relatively low stiffness. The layers are in ideal mutual contact.

The generalized plane temperature problem is reduced to determining the stress-strain state of the filler, which occupies a region in the form of a rectangular plate $|x'| \leqslant L$, $|y'| \leqslant H$ ($2L$ is the length of the plate and $2H$ is the thickness) based on the equations of the mixed problem of elasticity theory.

Generally speaking, the plate is bounded along the ν axis, perpendicular to the (x, y) plane.

For the deformation along the ν axis we have

$$\varepsilon_\nu = \lambda_0 T \tag{1.1}$$

where λ_0 is the coefficient of thermal expansion of the outer layers and T = const is the temperature increment.

Henceforth we will use dimensionless Cartesian coordinates x, y, referred to L. Then, the side surface of the filler is described by the equation $y = \pm h$, and its ends are described by the equation $x = \pm 1$.

We will write the relation between the stresses σ_x, σ_y, σ_{xy} and the strains ε_x, ε_y, $\varepsilon_{xy}/2$ of the transversely isotropic material for the case of generalized plane deformation, taking (1.1) into account, in the form

$$E\varepsilon_x = \sigma_x - \nu_0\sigma_y + E(1+\nu)\Delta\lambda T + E\lambda_0 T$$

$$E\varepsilon_y = \omega\sigma_y - \nu_0\sigma_x + E[\lambda_y + \nu_0(1-\nu)\Delta\lambda]T, \quad E\varepsilon_{xy} = \gamma_0\sigma_{xy} \tag{1.2}$$

$$E = \frac{E_x}{1-\nu^2}, \quad \omega = \frac{k-(k\nu')^2}{1-\nu^2}, \quad \nu_0 = \frac{k\nu'}{1-\nu}, \quad \gamma_0 = \frac{\gamma}{1-\nu^2}, \quad \gamma = \frac{E_x}{G}, \quad \Delta\lambda = \lambda_x - \lambda_0$$

Here we have assumed that the isotropy axis of the material is directed along the y axis, E_x and E_y are the elasticity moduli along the x and y axes, G is the shear modulus, ν and ν' are Poisson's ratios, and λ_x and λ_y are the coefficients of thermal expansion along the x and y axes respectively.

The stress σ_ν is found from (1.1).

We will write the equations of the anisotropic theory of elasticity, taking relations (1.1) and (1.2) into account, in the form

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$$\omega\frac{\partial^4 F}{\partial x^4}+\mu\frac{\partial^4 F}{\partial x^2\partial y^2}+\frac{\partial^4 F}{\partial y^4}=0,\quad \sigma_x=\frac{\partial^2 F}{\partial y^2},\quad \sigma_y=\frac{\partial^2 F}{\partial x^2},\quad \sigma_{xy}=-\frac{\partial^2 F}{\partial x\partial y},\quad \mu=\gamma_0-2\nu_0 \tag{1.3}$$

The matching conditions of the layers have the form

$$y=\pm h:\quad \varepsilon_x=\lambda_0 T,\quad \partial W/\partial x=0 \tag{1.4}$$

where W is the dimensionless displacement (referred to L) along the y axis.

The last relation in (1.4) reduces to the equality

$$y=\pm h:\quad \int_0^x\frac{\partial\sigma_x}{\partial y}dx=(\mu+\nu_0)\sigma_{xy}$$

There are no loads on the ends of the inner layer, i.e.

$$x=\pm 1:\quad \sigma_x=\sigma_{xy}=0 \tag{1.5}$$

We will write the condition for there to be no external actions on the external layers in the form

$$\int_0^1\sigma_y(x,\xi)\,dx=0 \tag{1.6}$$

where $\xi\in[-h,h]$ is a fixed coordinate along the y axis.

In view of the symmetry of the problem with respect to y we will consider the region $0\leqslant y\leqslant h$.

The solution of problem (1.1)–(1.6) presumably has a singularity (infinity) at the corner points ($x=\pm 1, y=h$) of the inner layer. We will obtain the finite smooth stress-strain state of the layer, which is identical with the solution of system (1.1)–(1.6) everywhere, with the exception of a certain small region of the corner points ($x=\pm 1, y=h$).

The latter problem in formulated as follows.

We distinguish a thin layer $h_2\ll h$: $h-h_2\leqslant y\leqslant h$ in the filler, where the surface $y=h$ is the contact boundary with the rigid outer layer.

In the thin layer $h-h_2\leqslant y\leqslant h$ it is required to find an exact solution of Eq.(1.3), corresponding to the internal mixed temperature problem, i.e. its solution must satisfy only the boundary conditions on the side surfaces $y=h-h_2$ and $y=h$. This layer will be called the boundary layer.

In the layer $0\leqslant y\leqslant h_1=h-h_2$ it is required to obtain an exact solution of the temperature problem, which enables the corresponding boundary conditions on the side surfaces $y=0, y=h_1$ and at the ends $x=\pm 1$, to be satisfied.

In the boundary layer of the filler $h_1\leqslant y\leqslant h$, we have

$$\begin{aligned}
&\sigma_x^{(2)}(x,y)=B_6[3x^2\eta h_2-\mu(\eta h_2)^3]+B_4[x^2-1-\mu(\eta h_2)^2]-\\
&-B_5\omega(\eta h_2)^2+B_3\eta h_2+B_2+\\
&+\sum_{m=1}\cos(\pi m x)\left\{\sum_{i=1}^{2}[R_{2i-1,m}\,\mathrm{sh}(\alpha_{i,m}\eta\delta)+R_{2i,m}\,\mathrm{ch}(\alpha_{i,m}\eta\delta)]\varkappa_i^2\right\}\\
&\sigma_y^{(2)}(x,y)=B_6(\eta h_2)^3+B_4(\eta h_2)^2+B_5\left(x^2-\frac{1}{3}\right)-\\
&-\sum_{m=1}\cos(\pi m x)\left\{\sum_{i=1}^{2}[R_{2i-1,m}\,\mathrm{sh}(\alpha_{i,m}\eta\delta)+R_{2i,m}\,\mathrm{ch}(\alpha_{i,m}\eta\delta)]\right\}\\
&\sigma_{xy}^{(2)}(x,y)=-3B_6x(\eta h_2)^2-2B_4x\eta h_2+\\
&+\sum_{m=1}\sin(\pi m x)\left\{\sum_{i=1}^{2}[R_{2i-1,m}\,\mathrm{ch}(\alpha_{i,m}\eta\delta)+R_{2i,m}\,\mathrm{sh}(\alpha_{i,m}\eta\delta)]\varkappa_i\right\}\\
&\delta=\frac{h_2}{h_1},\quad \eta=\frac{y-h_1}{h_2},\quad \alpha_{i,m}=\pi m\varkappa_i h_1,\quad \varkappa_i=\frac{1}{2}\left[(\mu\pm(\mu^2-4\omega)^{1/2}\right]^{1/2},\quad i=1,2
\end{aligned} \tag{1.7}$$

For the transversely isotropic material with pronounced anisotropy we can assume that $\mu>2\sqrt{\omega}$.

In the layer $0\leqslant y\leqslant h_1$ the solution of the temperature problem will be sought in the form

$$\sigma_x^{(1)}(x,y) = M_6(6x^2z^2 - \mu h_1^2 z^4) + N_6[x^4 - 1 - \omega(h_1 z)^4] + M_4[x^2 - 1 - \mu(h_1 z)^2] -$$
$$-N_4\omega(h_1 z)^2 + M_2 + \sum_{m=1} \cos(\pi m x)\left\{\sum_{i=1}^{2} C_{i,m}\,\mathrm{ch}(\alpha_{i,m}z)\varkappa_i^2\right\} -$$
$$-\sum_{n=1} \cos(\pi n z)\left\{\sum_{i=1}^{2} D_{i,n}\,\mathrm{ch}(\beta_{i,n}x)\right\}$$
$$\sigma_y^{(1)}(x,y) = M_6\left[(h_1 z)^2 - \left(x^4 - \frac{1}{5}\right)\frac{1}{\omega}h_1^{-2}\right] + N_6\left[6x^2(h_1 z)^2 - \frac{\mu}{\omega}\left(x^4 - \frac{1}{5}\right)\right] +$$
$$+ M_4(h_1 z)^2 + N_4\left(x^2 - \frac{1}{3}\right) + N_2 - \sum_{m=1} \cos(\pi m x)\left\{\sum_{i=1}^{2} C_{i,m}\,\mathrm{ch}(\alpha_{i,m}z)\right\} + \tag{1.8}$$
$$+\sum_{n=1} \cos(\pi n z)\left\{\sum_{i=1}^{2} D_{i,n}\,\mathrm{ch}(\beta_{i,n}x)\varkappa_i^2\right\}\frac{1}{\omega}$$
$$\sigma_{xy}^{(1)}(x,y) = -2h_1 x z[2(M_6 z^2 + N_6 x^2) + M_4] + \sum_{m=1} \sin(\pi m x)\left\{\sum_{i=1}^{2} C_{i,m}\,\mathrm{sh}(\alpha_{i,m}z)\varkappa_i\right\} +$$
$$+\sum_{n=1} \sin(\pi n z)\left\{\sum_{i=1}^{2} D_{i,n}\,\mathrm{sh}(\beta_{i,n}x)\varkappa_i\right\}\frac{1}{\sqrt{\omega}}$$
$$z = \frac{y}{h_1},\ \beta_{i,n} = \frac{\pi n \varkappa_i}{h_1\sqrt{\omega}},\ i = 1,2$$

In formulae (1.7) and (1.8) $C_{i,m}, D_{i,n}$ $(i = 1, 2)$, $R_{j,m}$ $(j = 1, 4)$, M_{2j}, N_{2j} $(j = 1, 2, 3)$, $B_j (j = 2, 3, 4, 5, 6)$ are constants, to be determined when solving the problem.

If the upper limit on the summation sign is not indicated, it must be assumed to be equal to infinity.

We will write the matching conditions for the boundary layer and the layer $0 \leq y \leq h_1$ in the form

$$\sigma_x^{(1)}(x,h_1) = \sigma_x^{(2)}(x,h_1),\quad \sigma_y^{(1)}(x,h_1) = \sigma_y^{(2)}(x,h_1)$$
$$\sigma_{xy}^{(1)}(x,h_1) = \sigma_{xy}^{(2)}(x,h_1),\quad \frac{\partial W^{(1)}}{\partial x}(x,h_1) = \frac{\partial W^{(2)}}{\partial x}(x,h_1) \tag{1.9}$$

where $W^{(2)}$ is the dimensionless displacement of points of the boundary layer along the y axis.

We reduce the last relation, taking the third relation of (1.9) into account, to the equation

$$y = h_1:\ \int_0^x \frac{\partial\sigma_x^{(1)}}{\partial y}dx = \int_0^x \frac{\partial\sigma_x^{(2)}}{\partial y}dx \tag{1.10}$$

In view of the symmetry of the stresses with respect to the x coordinate, instead of (1.5) we have

$$x = 1:\ \sigma_x^{(1)} = 0,\ \sigma_{xy}^{(1)} = 0 \tag{1.11}$$

Boundary conditions (1.4) on the surface $y = h$ have the form

$$y = h:\ \varepsilon_x^{(2)} = \lambda_0 T,\ \int_0^x \frac{\partial\sigma_x^{(2)}}{\partial y}dx = (\mu + \nu_0)\sigma_{xy}^{(2)} \tag{1.12}$$

It follows from (1.8) that condition (1.6) is automatically satisfied. The constants written above are found from Eqs (1.9)–(1.12). The presence of particular solutions in the polynomials of the partial differential equations (1.3) enables the convergence of series (1.7) and (1.8) to be accelerated.

We will briefly indicate a method of setting up an infinite system of algebraic equations for determining the required constants [1].

We use the following expansions of the function in Fourier series

$$\text{ch}(\alpha_{i,n}z)=\frac{1}{2}(\alpha_{i,n}\,\text{sh}\,\alpha_{i,n})z^2+\sum_{m=0} a^i_{n,m}\cos(\pi mz),\ i=1,2$$
$$\text{ch}(\beta_{i,n}x)=\frac{1}{2}(\beta_{i,n}\,\text{sh}\,\beta_{i,n})x^2+\sum_{m=0} b^i_{n,m}\cos(\pi mx),\ i=1,2$$
$$2x^2-x^4=\frac{7}{15}+48\sum_{m=1}(-1)^m(\pi m)^{-4}\cos(\pi mx) \tag{1.13}$$
$$x-x^3=-12\sum_{m=1}(-1)^m(\pi m)^{-3}\sin(\pi mx)$$

The order of decay of the Fourier coefficients $a^i_{n,m}$, $b^i_{n,m}$ is $(\pi m)^{-4}$.

The first and second equations of (1.9) and Eq.(1.5) are expanded in basis functions of x^2, 1, and cos (πmx). For example, as it applies to the first equation of (1.9), this means that, in this equation the hyperbolic functions and polynomial $2x^2-x^4$ are expanded in Fourier series using formulae (1.13). We then equate to zero the algebraic expressions with the factors cos (πmx) and x^2, and also the sum of the free terms of these Fourier expansions.

Similarly, the first equation of (1.11) is expanded in basis functions of z^2, 1 and cos (πnz).

Another group of infinite algebraic equations is obtained when the third relation of (1.9), relation (1.10) and the second relation of (1.12) are expanded in basis functions x and sin (πmx). Here it is sufficient to expand the polynomial $x-x^3$ in a Fourier series. In a similar way we can expand the second condition of (1.11) in functions z and sin (πnz).

The stress σ_y is found from Eq. (1.1).

2. THE RESULTS OF CALCULATIONS

All the calculations of the dimensionless stresses (referred to $E\ \Delta\lambda T$), presented in this section, were carried out for $k=3$, $\gamma=6$, $\nu=0.2$, $\nu'=0.1$, $h=0.2$, $h_2=h/6$ and $L_m=120$ and $L_n=15$, where L_m and L_n are the numbers at which the Fourier series in x and y respectively are terminated.

Figure 1 shows the distribution of the dimensionless stresses p_x (the continuous curves), p_y (the dash-dot curve), and p_{xy} (the dashed curves) along x in various sections of the layer. Curves 1, 2, 3 and 4 correspond to the sections $y=0$, $y=h_1/2$, $y=h_1$ and $y=h(h_1=h-h_2)$.

The table shows values of the dimensionless stresses $p_y\times 10^3$ at certain characteristic points of the inner layer.

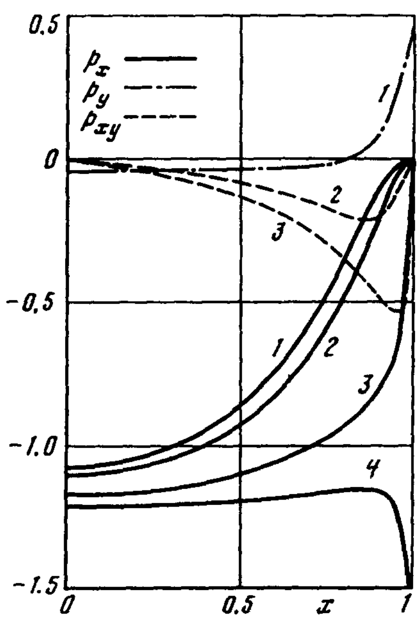

Fig. 1.

y	$x=0$	0.5	0.8	0.9	1
0	-39	-33	-5	83	451
$h_1/2$	-34	-22	9	48	369
h_1	-21	10	77	56	-656
h	-15	25	114	104	-1070

Below we give the results of a calculation of these stresses $p_x^{(1)}, p_y^{(1)}, p_{xy}^{(1)}$, and $p_x^{(2)}, p_y^{(2)}, p_{xy}^{(2)}$, on the surface $y = h_1$ where these two solutions are matched

x	0	0.5	0.8	0.9
$p_x^{(1)} \times 10^3$	-1170	-1100	-930	-814
$p_x^{(2)} \times 10^3$	-1170	-1100	-930	-814
$p_y^{(1)} \times 10^4$	-208	102	768	559
$p_y^{(2)} \times 10^4$	-213	97	763	557
$p_{xy}^{(1)} \times 10^3$	0	-132	-353	-486
$p_{xy}^{(2)} \times 10^3$	0	-131	-350	-481

These results confirm the high degree of convergence of the Fourier series in solution (1.7), (1.8).

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